Cavity tuning and field flattening. What's it all about?

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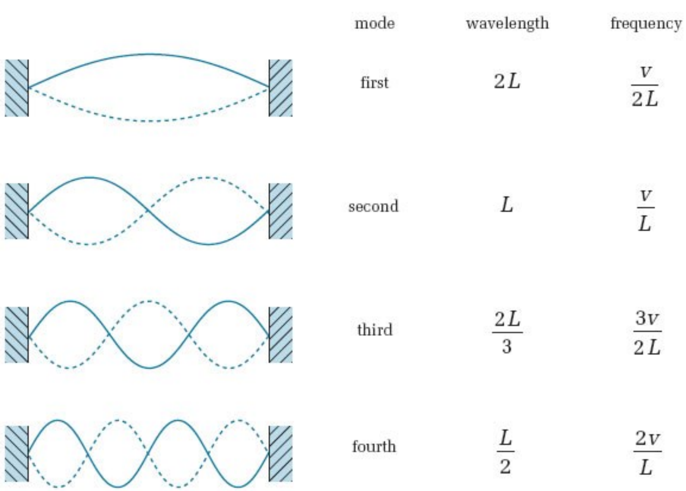
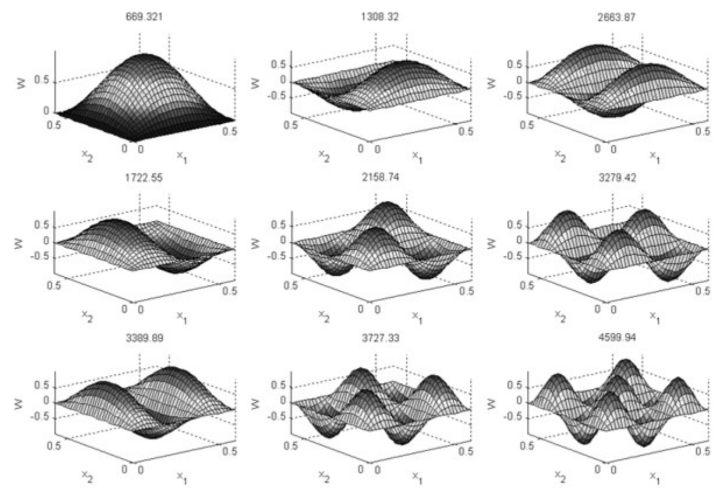
Cavity modes

When we talk about the ‘frequency’ of a cavity we’re referring to the frequency at which it resonates electromagnetically.

However cavities can resonate in many different *modes* just like a string or a flat plate





Transverse Electric & Transverse Magnetic modes

Most accelerator cavities have some sort of cylindrical symmetry and two types of resonant modes exist.

Transverse electric modes have the electric field transverse to the symmetry axis (which is usually the beam direction) and are designated TE_{mnp}.
Transverse magnetic modes have the magnetic field transverse to the symmetry axis and are designated TM_{mnp}.

The subscripts mean:

m (= 0, 1, 2, …) – the number of full period variations of the field components in θ.

n (= 1, 2, 3, …) – the number of zeroes in the axial field in the radial direction.

p (= 0, 1, 2, …) – the number of half period variations of the field along the z axis.

For a simple cylindrical cavity analytical solutions exist for all the TE and TM modes.

The general solution for the field components of the TM_{mnp} mode in a cylindrical cavity is:

$$E_z = E_0 J_m(k_{mn} r)\cos m\theta \cos\frac{p\pi z}{l_c} e^{j\omega t}$$

$$E_r = -\frac{p\pi}{l_c}\frac{a}{x_{mn}} E_0 J'_m(k_{mn} r)\cos m\theta \sin\frac{p\pi z}{l_c} e^{j\omega t}$$

$$E_\theta = -\frac{p\pi}{l_c}\frac{ma^2}{x_{mn}^2 r} E_0 J_m(k_{mn} r)\sin m\theta \sin\frac{p\pi z}{l_c} e^{j\omega t}$$

$$B_z = 0$$

$$B_r = -j\omega\frac{ma^2}{x_{mn}^2 r c^2} E_0 J_m(k_{mn} r)\sin m\theta \cos\frac{p\pi z}{l_c} e^{j\omega t}$$

$$B_\theta = -j\omega\frac{a}{x_{mn} c^2} E_0 J'_m(k_{mn} r)\cos m\theta \cos\frac{p\pi z}{l_c} e^{j\omega t}$$

The general solution for the field components of the TE_{mnp} mode in a cylindrical cavity is:

$$B_z = B_0 J_m(k_{mn} r)\cos m\theta \cos\frac{p\pi z}{l_c} e^{j\omega t}$$

$$B_r = \frac{p\pi}{l_c}\frac{a}{x'_{mn}} B_0 J'_m(k_{mn} r)\cos m\theta \cos\frac{p\pi z}{l_c} e^{j\omega t}$$

$$B_\theta = -\frac{p\pi}{l_c}\frac{ma^2}{x'^2_{mn} r} B_0 J_m(k_{mn} r)\sin m\theta \cos\frac{p\pi z}{l_c} e^{j\omega t}$$

$$E_z = 0$$

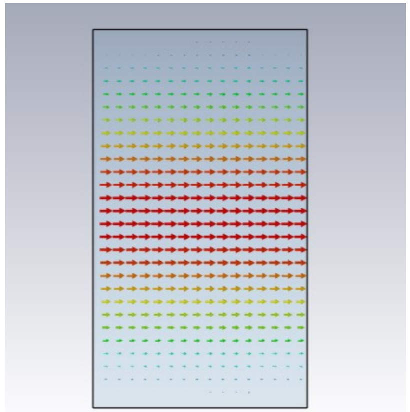
$$E_r = j\omega\frac{ma^2}{x'^2_{mn} r} B_0 J_m(k_{mn} r)\sin m\theta \sin\frac{p\pi z}{l_c} e^{j\omega t}$$

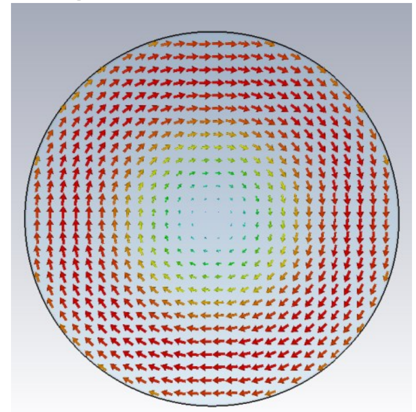
$$E_\theta = j\omega\frac{a}{x'_{mn}} B_0 J'_m(k_{mn} r)\cos m\theta \sin\frac{p\pi z}{l_c} e^{j\omega t}$$

Some useful cylindrical modes

Electric field | Magnetic field

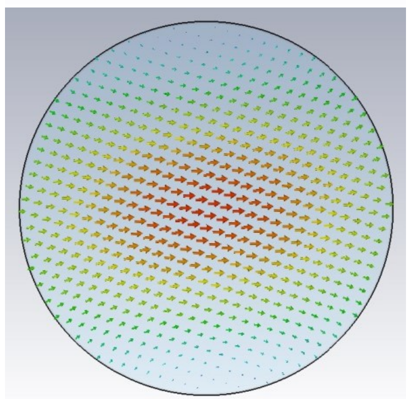
TM_{010}

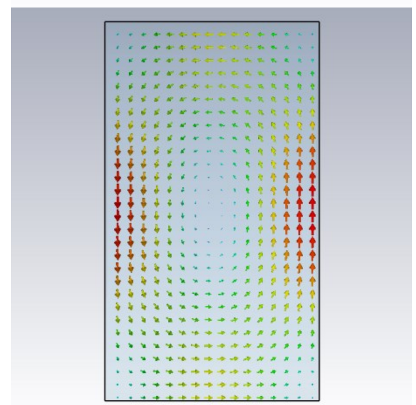




Accelerating mode

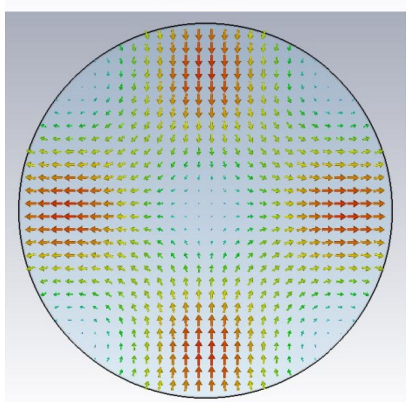
TE_{111}

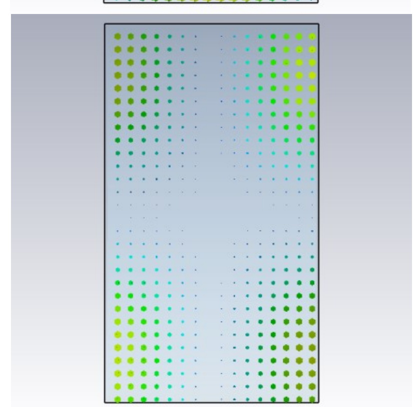




Dipole mode

TE_{211}





Quadrupole mode

Longitudinal modes

For every 'transverse' mode – TE/M_{mn} – there are an infinite number of 'longitudinal' modes where the electric field components have the form

$\sin\frac{p\pi z}{l_c}$ for TE modes

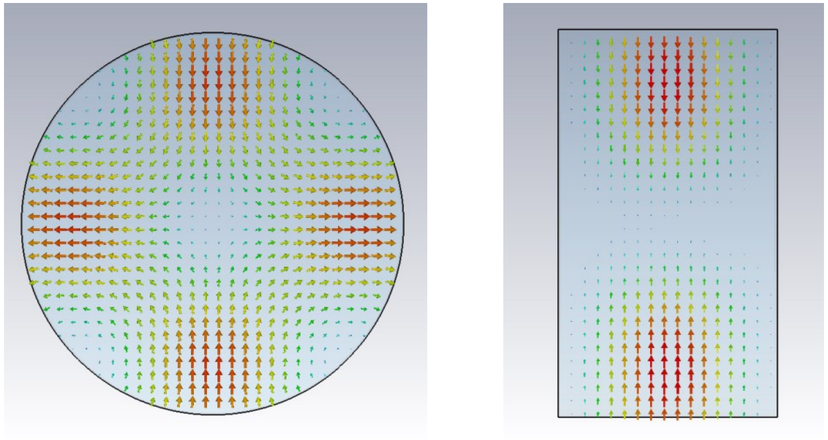
And

$\cos\frac{p\pi z}{l_c}$ for TM modes

RFQ modes

The mode of interest in an RFQ is the TE_{210} quadrupole mode.

Note however that the boundary conditions of a simple cavity forbid the existence of a TE_{mn0} mode, the lowest transverse electric mode being TE_{mn1}.



Electric field of the lowest quadrupole mode in a cylindrical cavity – TE_{211}.

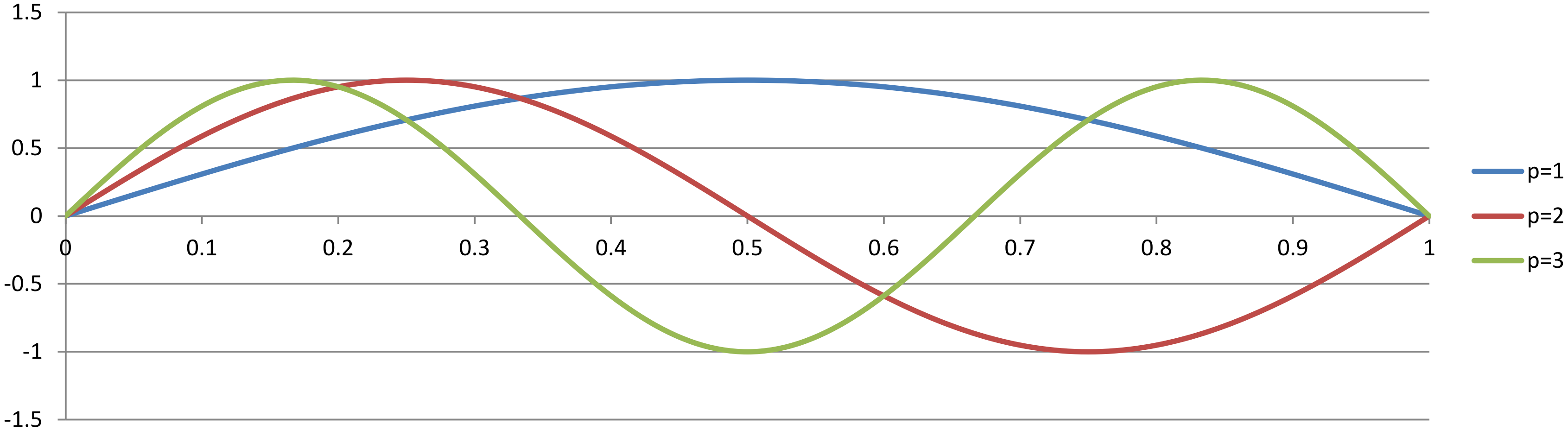
In order to achieve a TE_{210}-like mode - a ‘flat’ field - the ends of the RFQ have to be modified.

With the modified end regions the longitudinal modes have a $\cos\frac{p\pi z}{l_c}$ variation more like a TM mode.

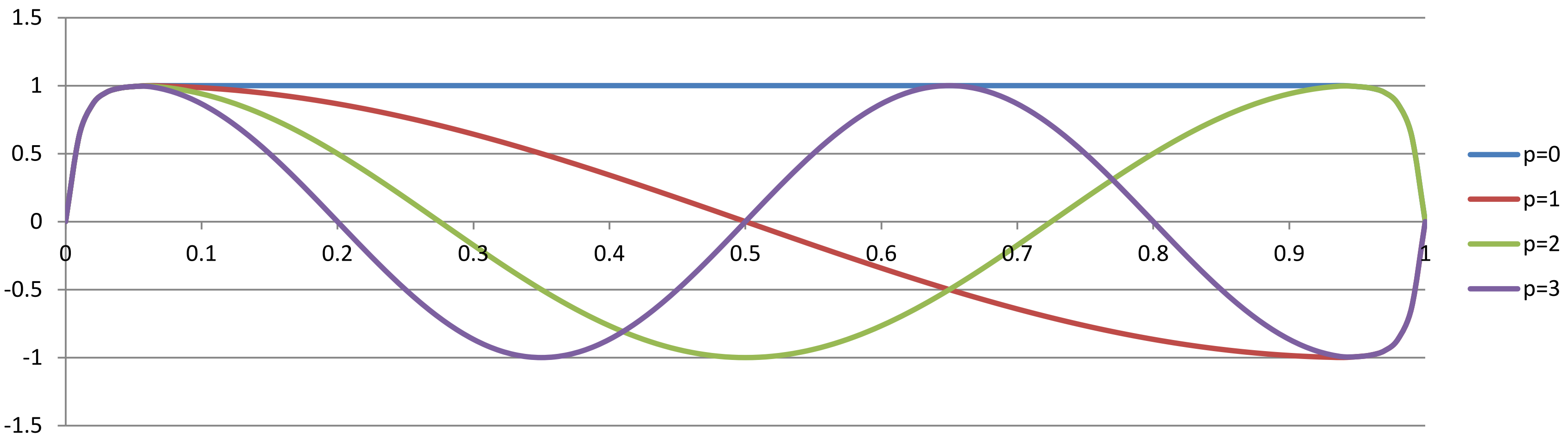
Modifying the ends allows all the TE_{mn0} modes to exists (in particular the TE_{110} dipole mode(s)).

TE_{mn0}-like modes

Longitudinal modes of TE fields in a simple cavity



Longitudinal modes of TE fields in a modified cavity with a TE_{mn0}-like mode



So what does all this mean?

In an ideal cavity a ‘pure’ mode can be excited by tuning to exactly the resonant frequency of that mode.

In a real cavity, which may contain imperfections, the field shape is a superposition of many (in principle an infinite number of) different modes.

The amount of an unwanted mode that is present depends on the magnitude of local frequency errors and the difference in frequency between the wanted and unwanted modes.

In an RFQ the wanted mode is TE_{210}.
The unwanted modes are $TE_{21(1,2,3,\ldots)}$ which lead to a non-flat quadrupole field and $TE_{11(0,1,2,3,\ldots)}$ which introduce dipole components.

Fields in an RFQ

The fields in an RFQ can be expressed as

$$E(z) = \sum_{p=0}^{N} \left[A(p) * TE_{21p} + B(p) * TE^{1}_{11p} + C(p) * TE^{2}_{11p} \right]$$

Where TE^{1}_{11p} and TE^{2}_{11p} are the two orientations of the dipole modes.

The purpose of the tuning and flattening exercise is to achieve

$$A(0) = 1$$
$$A(p)|_{p=1..N} = 0$$
$$B(p)|_{p=0..N} = 0$$
$$C(p)|_{p=0..N} = 0$$

Fields in each quadrant

More explicitly, numbering each quadrant q=1, 2, 3 or 4:

$$E(z,q)=\sum_{p=0}^{N}\left[A(p)Q(p)(-1)^{q-1}f(z,p)\pm B(p)D_1(p)f(z,p)\pm C(p)D_2(p)f(z,p)\right] \quad (1)$$

Q(p), $D_1(p)$ and $D_2(p)$ are normalisation factors to equalise the stored energy in each mode.

D_1 is +ve if q = 1, -ve if q = 3, 0 if q = 2 or 4.

D_2 is +ve if q = 2, -ve if q = 4, 0 if q = 1 or 3.

With ω_{q0} the angular frequency of the desired quadrupole mode and ω_{qp}, ω_{d1p}, ω_{d2p} the frequencies of the other modes

$$f(z,p)=\cos\left(\frac{2\pi(z-z_0)}{\lambda_p}\right)\text{if } \omega_p>\omega_{q0}$$

$$f(z,p)=\cosh\left(\frac{2\pi(z-z_0)}{i\lambda_p}\right)\text{if } \omega_p<\omega_{q0}$$

$$z_0=\frac{L}{2}-\lambda_p\frac{p}{4}$$

$$\frac{1}{\lambda_p}=\frac{1}{2\pi c}\sqrt{\omega_p^2-\omega_{q0}^2}$$

<u>The values of *A(p)*, *B(p)* & *C(p)* can be determined by a bead-pull of each quadrant.</u>

Fields due to frequency perturbations

A frequency perturbation $\omega = \omega_k$ at $z = k$ introduces a component of mode p given by

$$\text{Amount of mode } p = \frac{\omega_k^2 - \omega_0^2}{\omega_p^2 - \omega_0^2} * \left|\text{mode } p\right|_{z=k}$$

Generalising this to the quadrants of the RFQ leads to

$$A(p) = -\frac{1}{L}\int_0^L \left(\frac{\omega_z^2 - \omega_0^2}{\omega_{qp}^2 - \omega_0^2} f(z,p) \right) .dz \qquad (2)$$

$$B(p) = -\frac{1}{L}\int_0^L \left(\frac{\omega_z^2 - \omega_0^2}{\omega_{d1p}^2 - \omega_0^2} f(z,p) \right) .dz \qquad (3)$$

$$C(p) = -\frac{1}{L}\int_0^L \left(\frac{\omega_z^2 - \omega_0^2}{\omega_{d2p}^2 - \omega_0^2} f(z,p) \right) .dz \qquad (4)$$

If the ω_z are frequency shifts at fixed tuner positions, equations (1), (2), (3) & (4) can be solved in a least squares sense for ω_z. This along with the constraint of fixed ω_{q0} leads to new tuner positions to give $A(p)_{p>0}$, $B(p)$ & $C(p) = 0$.